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On Relative Integral Bases for Cyclic Quartic Fields

JOHN A. HYMO AND CHARLES J. PARRY

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

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When does a cyclic quartic field have an integral basis over its quadratic subfield? A simple, easy to use answer is given to this question. Moreover, a basis is given whenever it exists. © 1990 Academic Press, Inc.

1. INTRODUCTION

A necessary and sufficient condition is given for a cyclic quartic field to have an integral basis over its quadratic subfield. An explicit integral basis is given for this relative extension whenever it exists. For a fixed quadratic field k, it is shown that $1/g$ of all cyclic quartic fields which contain k have a relative integral basis. Here g denotes the 2-rank of the ideal class group of k.

In [2], Edgar and Peterson give a criterion for a cyclic quartic field to have a relative integral basis over its quadratic subfield. However, their criterion is not explicit and they give no bases.

We use the description of cyclic quartic fields given in [3]. A criterion of Mann [5] is used to determine when the extension has an integral basis. In [1], Bird and Parry have obtained similar results for bicyclic biquadratic fields over their quadratic subfields.

2. NOTATION

We adopt the following notations:

L/M: An extension of number fields.

$\Delta_{L/M}$: Discriminant of L/M.

K: A cyclic quartic extension of Q.

$k = Q(\sqrt{D})$: Quadratic subfield of K.

$\varepsilon_0 = r + t\sqrt{D}$: Fundamental unit of k.

$\delta = 0$ if $r, t \in Z$; 2 if $r, t \notin Z$.

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In [3], it is shown that $K = Q(\sqrt{A(D+B\sqrt{D})})$, where A is square free and odd, $B > 0$, and $C > 0$ with $D = B^2 + C^2$ square free and $A, B, C \in Z$. Moreover, $(A, D) = 1$.

3. Existence of an Integral Basis

We first determine the relative discriminant of K/k.

LEMMA 1. *The relative discriminant of K/k is given by $\Delta_{K/k} = (\Delta)$, where*

$$\Delta = \begin{cases} 4A\sqrt{D} & \text{if } D \equiv 0 \pmod 2 \text{ or } D \equiv 1 \pmod 4, \\ & \quad B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4 \\ A\sqrt{D} & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4 \\ 8A\sqrt{D} & \text{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2. \end{cases}$$

Proof. By [3]

$$\Delta_{K/Q} = \begin{cases} 2^8A^2D^3 & \text{if } D \equiv 0 \pmod 2 \\ 2^4A^2D^3 & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4 \\ A^2D^3 & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4 \\ 2^6A^2D^3 & \text{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2. \end{cases}$$

Since

$$\Delta_{k/Q} = \begin{cases} D & \text{if } D \equiv 1 \pmod 4 \\ 4D & \text{if } D \not\equiv 1 \pmod 4 \end{cases}$$

and

$$\Delta_{K/Q} = N_{k/Q}(\Delta_{K/k})\Delta_{k/Q}^2 = \Delta_{K/k}^2 \Delta_{k/Q}^2,$$

the lemma follows.

LEMMA 2. *K/k has an integral basis if and only if $K = k(\sqrt{A'\varepsilon\sqrt{D}})$, where*

$$A' = \begin{cases} 2A & \text{if } D \equiv 1 \pmod 4 \text{ and } B \equiv 1 \pmod 2 \\ A & \text{otherwise} \end{cases}$$

and ε is a unit of k with norm -1. In fact, we may choose $\varepsilon = \varepsilon_0$.

Proof. Mann [5] shows that K/k has an integral basis if and only if $K=k(\sqrt{\Delta})$ for some generator Δ of $\Delta_{K/k}$. By Lemma 1, this is equivalent to $K=k(\sqrt{A'\varepsilon\sqrt{D}})=k(\sqrt{2^a A\varepsilon\sqrt{D}})$ for some unit ε of k, where $a=1$ when B and D are both odd and $a=0$ otherwise.

But $K=k(\sqrt{A(D+B(\sqrt{D}))})=k(\sqrt{2^a A\varepsilon\sqrt{D}})$ if and only if $A(D+B\sqrt{D})=2^a A\varepsilon\sqrt{D}\,s^2$ for some $s\in k$. Equivalently, $B+\sqrt{D}=2^a\varepsilon s^2$. Taking norms gives $-C^2=B^2-D=(2^a)^2 N_{k/Q}(\varepsilon)(s\bar{s})^2$, so ε has norm -1.

Since $k(\sqrt{A(D+B\sqrt{D})})=k(\sqrt{2^a A\varepsilon\sqrt{D}})$, $\varepsilon>0$, so $\varepsilon=\varepsilon_0^i$ for some integer i. Since ε has norm -1, i must be odd, so $k(\sqrt{2^a A\varepsilon\sqrt{D}})=k(\sqrt{2^a A\varepsilon_0\sqrt{D}})$.

For the remainder of the article, we will assume, unless otherwise stated, that $N(\varepsilon_0)=-1$. Recall that $\varepsilon_0=r+t\sqrt{D}$ and $2^{\delta/2}r$, $2^{\delta/2}t$ are integers with $\delta=0$ or 2.

LEMMA 3. *If* $2^{\delta/2}(r+i)=(u+vi)^2\,(X+Yi)$ *in* $Z[i]$ *with* $X+Yi$ *square free and* $X>0$ *then*

$$k(\sqrt{\varepsilon_0\sqrt{D}})=\begin{cases} k(\sqrt{D+X\sqrt{D}}) & \text{if } \delta=0\\ k(\sqrt{D+|Y|\sqrt{D}}) & \text{if } \delta=2.\end{cases}$$

Moreover, when $D\equiv 1 \pmod 4$, $X\equiv\delta/2 \pmod 2$ *and* $Y\equiv\delta/2+1 \pmod 2$.

Proof. Note that

$$\begin{aligned} k(\sqrt{\varepsilon_0\sqrt{D}})&=k(\sqrt{tD+r\sqrt{D}})=k(\sqrt{2^\delta t(2^\delta t^2D+2^{\delta/2}r\sqrt{2^\delta t^2 D})})\\ &=k(\sqrt{A_1(D_1+B_1\sqrt{D_1})}),\end{aligned}$$

where $A_1=2^\delta t$, $B_1=2^{\delta/2}r$, and $D_1=2^\delta t^2 D$. Also, $D_1-B_1^2=2^\delta t^2 D-2^\delta r^2=2^\delta=C_1^2$, where $C_1=2^{\delta/2}$. Since $B_1+C_1 i=2^{\delta/2}(r+i)=(u+v^i)^2\,(X+Y^i)$, it follows that $2^\delta t^2 D=D_1=B_1^2+C_1^2=(u^2+v^2)^2\,(X^2+Y^2)$. Since $X+Yi$ is square free and has no rational factor (because $2^{\delta/2}(r+i)$ has none), it follows that X^2+Y^2 is also square free. Thus $2^{\delta/2}t=u^2+v^2$ and $X^2+Y^2=D$. Moreover, $B_1=X(u^2-v^2)-Y(2uv)$, $C_1=X(2uv)+Y(u^2-v^2)$, and $D_1=(u^2+v^2)^2 D$. It follows from [3, p. 5–6] that

$$\begin{aligned} k(\sqrt{\varepsilon_0\sqrt{D}})&=k(\sqrt{A_1(D_1+B_1\sqrt{D_1})})=k(\sqrt{A_1(u^2+v^2)(D+X\sqrt{D})})\\ &=k(\sqrt{2^\delta t^2 2^{\delta/2}(D+X\sqrt{D})})=k(\sqrt{2^{\delta/2}(D+X\sqrt{D})})\\ &=\begin{cases} k(\sqrt{D+X\sqrt{D}}) & \text{if } \delta=0\\ k(\sqrt{D+|Y|\sqrt{D}}) & \text{if } \delta=2.\end{cases}\end{aligned}$$

If D is odd, then $-2^{\delta}=N(2^{\delta/2}\cdot\varepsilon_0)=(2^{\delta/2}r)^2-(2^{\delta/2}t)^2D$. Thus $2^{\delta/2}t=u^2+v^2$ is odd, and hence u^2-v^2 is also odd. Since $2^{\delta/2}=C_1=X(2uv)+Y(u^2-v^2)$, we see that $Y\equiv 2^{\delta/2}\equiv\delta/2+1 \pmod 2$. Thus $X\equiv X^2\equiv D-Y^2\equiv 1-Y\equiv\delta/2 \pmod 2$.

THEOREM 1. *If D is odd then K/k has an integral basis if and only if $B\pm Ci$ divides $2^{\delta/2}(r+i)$ in $Z[i]$. If D is even then K/k has an integral basis if and only if $r+i=(u+vi)^2\,(B\pm Ci)$ for some $u, v\in Z$.*

Proof. As in Lemma 3, write $2^{\delta/2}(r+i)=(u+vi)^2\,(X+Yi)$, where $X>0$. Let $Z=X$ or $|Y|$ according as $\delta=0$ or $\delta=2$ and set $Z'=\sqrt{D-Z^2}$.
If K/k has an integral basis then Lemmas 2 and 3 show that

$$K=k(\sqrt{A(D+B\sqrt{D})})=k(\sqrt{A'\varepsilon_0\sqrt{D}})=k(\sqrt{A'(D+Z\sqrt{D})}).$$

Unless both B and D are odd, Lemma 2 shows that $A'=A$. Theorem 1 of [3] shows that $B+Ci=i^{\delta/2}(X\pm Yi)$. Moreover, when D is even, $\delta=0$, so $B+Ci=X\pm Yi$. If D and B are both odd then $A'=2A$. It follows from Lemmas 2 and 3 and Theorem 2 of [3] that

$$k(\sqrt{A(D+B\sqrt{D})})=K=k(\sqrt{2A(D+Z\sqrt{D})})=k(\sqrt{A(D+Z'\sqrt{D})}).$$

Thus $B+Ci=(i)^{1-\delta/2}(X\pm Yi)$.

Conversely, assume D is odd and $B\pm Ci$ divides $2^{\delta/2}(r+i)$, so $2^{\delta/2}(r+i)=(x+yi)(B\pm Ci)$ for some $x, y\in Z$. Taking norms, we obtain $2^{\delta}t^2D=2^{\delta}(r^2+1)=(x^2+y^2)(B^2+C^2)=(x^2+y^2)D$, so $x^2+y^2=(2^{\delta/2}t)^2$. Since $2^{\delta/2}(r+i)$ is not divisible by a rational prime, neither is $x+yi$. Because its norm is a square, $x+yi=i^b(w+zi)^2$ with with $b=0$ or 1 and $w, z\in Z$. Thus

$$2^{\delta/2}(r+i)=i^b(w+zi)^2\,(B\pm Ci)=\begin{cases}(w+zi)^2\,(B\pm Ci) & \text{if } b=0\\ (w+zi)^2\,(\pm C+Bi) & \text{if } b=1.\end{cases}$$

Hence

$$2^{\delta/2}(r+i)\equiv\begin{cases}B+Ci \pmod 2 & \text{when } b=0\\ C+Bi \pmod 2 & \text{when } b=1.\end{cases}$$

Also, when $\delta=2$, note that $2^{\delta/2}(r+i)\equiv 1 \pmod 2$. If $\delta/2\not\equiv b \pmod 2$ then $B\equiv 1 \pmod 2$, so by Lemmas 1 and 3 and Theorem 2 of [3],

$$\begin{aligned}k(\sqrt{\Delta\varepsilon_0})&=k(\sqrt{2A\varepsilon_0\sqrt{D}})=k(\sqrt{2A(D+C\sqrt{D})})\\&=k(\sqrt{A(D+B\sqrt{D})})=K.\end{aligned}$$

When $\delta/2 \equiv b(\bmod 2)$, $B \equiv 0(\bmod 2)$, so $k(\sqrt{\Delta\varepsilon_0}) = k(\sqrt{A\varepsilon_0\sqrt{D}}) = k(\sqrt{A(D+B\sqrt{D})}) = K$. Thus Lemma 2 applies to show that K/k has an integral basis.

Finally, if D is even and $r+i=(u+vi)^2\,(B \pm Ci)$, then by Lemmas 1 and 3, $k(\sqrt{\Delta\varepsilon_0}) = k(\sqrt{A\varepsilon_0\sqrt{D}}) = k(\sqrt{A(D+B\sqrt{D})}) = K$. Thus Lemma 2 again applies.

4. An Integral Basis

In this section an explicit integral basis is given for K/k whenever it exists. Let $s = \frac{1}{2}(2r+1)$.

Theorem 2. *If D is odd and $K = k(\sqrt{2A\varepsilon_0\sqrt{D}})$ then $1, \sqrt{2A\varepsilon_0\sqrt{D}}$ is an integral basis for K/k. If $K = k(\sqrt{A\varepsilon_0\sqrt{D}})$ then an integral basis for K/k is given by*

(i) $1, \frac{1}{2}((1+(-1)^s\sqrt{D})/2 + \sqrt{A\varepsilon_0\sqrt{D}})$ *if* $D \equiv 1(\bmod 4)$, $A \equiv 3(\bmod 4)$ *and* $\delta = 2$;

(ii) $1, (1+\sqrt{A\varepsilon_0\sqrt{D}})/2$ *if* $D \equiv A + r \equiv 1(\bmod 4)$ *and* $\delta = 0$;

(iii) $1, \sqrt{A\varepsilon_0\sqrt{D}}$ *otherwise.*

Proof. First assume that D is odd and $K = k(\sqrt{2A\varepsilon_0\sqrt{D}}) = k(\sqrt{2A(D+Z\sqrt{D})})$, where by Lemma 3, $Z = X$ or $|Y|$ according as $\delta = 0$ or $\delta = 2$. By Theorem 2 of [3] $K = k(\sqrt{A(D+Z'\sqrt{D})})$ where $Z' = \sqrt{D-Z^2}$. By Lemma 3, Z is even, so Z' is odd. Hence, by Lemma 1, $\Delta_{K/k} = (8A\sqrt{D})$. Since the field basis $1, \sqrt{2A\varepsilon_0\sqrt{D}}$ has discriminant $8A\varepsilon_0\sqrt{D}$, it is an integral basis for K/k.

For the remainder of this proof, we have $K = k(\sqrt{A\varepsilon_0\sqrt{D}})$. In cases (i) and (ii), it is sufficient to show that the second element of the basis is an integer. In (i) this is equivalent to showing that

$$\frac{1+(-1)^s\sqrt{D}}{2} \equiv \sqrt{A\varepsilon_0\sqrt{D}}\ (\bmod 2)$$

or

$$\left(\frac{1+(-1)^s\sqrt{D}}{2}\right)^2 \equiv A\varepsilon_0\sqrt{D}\ (\bmod 4).$$

The last congruence can be restated in the form

$$\frac{1}{2}\left(\frac{1+D}{2}+(-1)^s\sqrt{D}\right)\equiv A(r+t\sqrt{D})\sqrt{D}\equiv\frac{A(2t)D+A(2r)\sqrt{D}}{2}\ (\mathrm{mod}\ 4).$$

This can be proved by showing $(1+D)/2\equiv A(2t)D(\mathrm{mod}\ 4)$, $(-1)^s\equiv A(2r)(\mathrm{mod}\ 4)$, and $(1+D)/2-(-1)^s\equiv A(2t)D-A(2r)(\mathrm{mod}\ 8)$. First note that since $2r$ and $2t$ are odd integers $-4=N(2\varepsilon_0)=(2r)^2-(2t)^2D\equiv 1-D(\mathrm{mod}\ 8)$, so $D\equiv 5(\mathrm{mod}\ 8)$. Also, from the proof of Lemma 3, $2t$ is the sum of two squares, so $2t\equiv 1(\mathrm{mod}\ 4)$. Therefore,

$$A(2t)D\equiv A\equiv 3\equiv\frac{1+D}{2}\ (\mathrm{mod}\ 4).$$

Also, since $s=\frac{1}{2}(2r+1)$, $(-1)^s\equiv -2r\equiv A(2r)(\mathrm{mod}\ 4)$.

The congruence values of $D(\mathrm{mod}\ 16)$ will be used to prove the final condition. If $D\equiv 5(\mathrm{mod}\ 16)$ then $-4=(2r)^2-D(2t)^2\equiv(2r)^2-5(2t)^2$ $(\mathrm{mod}\ 16)$. Since $2r$ and $2t$ are odd, their squares are 1 or $9(\mathrm{mod}\ 16)$. It follows that $(2r)^2\equiv(2t)^2$ $(\mathrm{mod}\ 16)$ and $2r\equiv\pm 2t(\mathrm{mod}\ 8)$. Hence,

$$A(2t)D-A(2r)\equiv A(10t-2r)\equiv\begin{cases}4(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv 2t(\mathrm{mod}\ 8)\\ 2(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv -2t(\mathrm{mod}\ 8).\end{cases}$$

But $(1+D)/2-(1)^s\equiv 3-(-1)^s$ $(\mathrm{mod}\ 8)$ has the same value.

Similarly, when $D\equiv 13(\mathrm{mod}\ 16)$, $-4\equiv(2r)^2-(2t)^2D\equiv(2r)^2+3(2t)^2$ $(\mathrm{mod}\ 16)$, so that $(2r)^2\equiv(2t)^2+8(\mathrm{mod}\ 16)$. Hence $2t\pm 2r=4z$ for some odd integer z. Thus

$$\begin{aligned}A(2t)D-A(2r)&\equiv(10t-2r)A\\&\equiv\begin{cases}(8t+4z)A\equiv 0(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv 2t(\mathrm{mod}\ 4)\\ (12t-4z)A\equiv 6(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv -2t(\mathrm{mod}\ 4).\end{cases}\end{aligned}$$

Also,

$$\frac{1+D}{2}-(-1)^s\equiv 7-(-1)^s\equiv\begin{cases}0(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv 2t(\mathrm{mod}\ 4)\\ 6(\mathrm{mod}\ 8) & \text{if}\quad 2r\equiv -2t(\mathrm{mod}\ 4).\end{cases}$$

In (ii), we first note that $-1=-r^2-t^2D$ implies r is even and t is odd. Since t is the sum of two squares, $t\equiv D\equiv 1(\mathrm{mod}\ 4)$. Thus $A\varepsilon_0\sqrt{D}=AtD+Ar\sqrt{D}\equiv A+r\equiv 1(\mathrm{mod}\ 4)$, so $\sqrt{A\varepsilon_0\sqrt{D}}\equiv 1(\mathrm{mod}\ 2)$ and hence $(1+\sqrt{A\varepsilon_0\sqrt{D}})/2$ is an integer. In (iii)we only need to show that $\Delta_{K/k}=(4A\sqrt{D})$. First, assume D is odd, $A+r\equiv 3(\mathrm{mod}\ 4)$ and $r, t\in Z$. From the proof of Lemma 3, $r=(u^2-v^2)X-2uvY$ with r, X and exactly one of u or

v even, so $r \equiv X \pmod 4$. Thus, $A + X \equiv A + r \equiv 3 \pmod 4$, so Lemmas 3 and 1 show $\Delta_{K/k} = (4A\sqrt{D})$. When D is odd with $A \equiv 1 \pmod 4$ and $r, t \notin Z$ then Y is even by Lemma 3. Since $-4 = (2r)^2 - (2t)^2 D = (2r)^2 - (2t)^2 (X^2 + Y^2) \equiv 1 - (Y^2 + 1) \equiv -Y^2 \pmod 8$, we have $Y \equiv 2 \pmod 4$. Thus $A + |Y| \equiv 3 \pmod 4$, so Lemmas 3 and 1 again show $\Delta_{K/k} = (4A\sqrt{D})$.

When D is even, Lemma 1 shows that $\Delta_{K/k} = (4A\sqrt{D})$.

5. Class Number Considerations

In this section a connection is established between the existence of an integral basis for K/k and the class number of k. In Theorem 3, a new proof of Theorem 1 of [2] is given, while Theorem 4 strengthens the result of Theorem 2 of [2].

THEOREM 3. *If k has odd class number then K/k always has an integral basis.*

Proof. If $h(k)$ is odd then $D = 2$ or $D = p \equiv 1 \pmod 4$ with p prime. Thus $N(\varepsilon_0) = -1$ and D can be expressed as a sum of two squares in a unique way. When $D = 2$, $B = 1$ and $\varepsilon_0 = 1 + \sqrt{2}$, so $k(\sqrt{A\varepsilon_0\sqrt{2}}) = k(\sqrt{A(2 + \sqrt{2})}) = K$. Hence by Lemma 2, K/k has an integral basis.

Assume now $D = p = B^2 + C^2$. Since $X^2 + Y^2 = D$ and the representation of D as a sum of two squares is unique, $B \pm Ci$ and $X + Yi$ are associates. It follows from Lemma 3 and Theorem 1 that K/k has an integral basis.

THEOREM 4. *Assume $h(k)$ is even and that $D = p_1 \cdots p_n$ $(n \geqslant 2)$ with $p_1, \ldots, p_n$ distinct primes, $p_i \equiv 1$ or $2 \pmod 4$. If $N(\varepsilon_0) = +1$ then K/k never has an integral basis. If $N(\varepsilon_0) = -1$ then the ratio ρ of cyclic quartic extensions K/Q, such that K/k has an integral basis to all cyclic quartic extensions K/Q which contain k, is $1/2^{n-1} = 1/2^g$, where g is the 2-rank of the ideal class group of k.*

Proof. If $N(\varepsilon_0) = +1$ then Lemma 2 shows K/k never has an integral basis.

Assume next that $N(\varepsilon_0) = -1$ and D is odd. Then $D = B^2 + C^2$, with $B \neq C$, can be expressed as the sum of two squares in 2^{n-1} ways, so there are 2^n distinct fields of the form $k(\sqrt{A(D + B\sqrt{D})})$ for any fixed odd integer A. If $X + Yi$ is determined as in Lemma 3, then by Theorem 1, only $k(\sqrt{A(D + X\sqrt{D})})$ and $k(\sqrt{A(D + |Y|\sqrt{D})})$ have integral basis over k. Thus $\rho = 2/2^n = 1/2^{n-1}$. Since $N(\varepsilon_0) = -1$, $n - 1 = g$ is the 2-rank of the ideal class group of k (see Satz 106 and 107 of [4].)

Assume now that $N(\varepsilon_0) = -1$ and D is even. Since $h(k)$ is even, $D \neq 2$. Here $D = B^2 + C^2$, with $B \neq C$, can be expressed as the sum of two squares in 2^{n-2} ways. Hence for any fixed odd integer A, there are 2^{n-1} fields of the form $k(\sqrt{A(D + B\sqrt{D})})$. By Theorem 1, there is only one field K such that K/k has an integral basis. Thus $\rho = 1/2^{n-1}$ and as above $g = n - 1$.

6. Examples

1. $D = 1105 = 5 \cdot 13 \cdot 17$; $\varepsilon_0 = 28488 + 857\sqrt{1105}$; $r + i = 28488 + i = (29 - 4i)^2 (32 + 9i)$. By Theorem 1, $K = k(\sqrt{A(1105 + B\sqrt{1105})})$ has an integral basis over k for $B = 9$ and 32, but does not have a basis for $B = 4$, 12, 23, 24, 31, and 33. For $B = 32$, $K = k(\sqrt{A\varepsilon_0\sqrt{D}})$ by Lemma 2, so by Theorem 2, a basis is

$$\begin{cases} 1, \dfrac{1 + \sqrt{A\varepsilon_0\sqrt{D}}}{2} & \text{if} \quad A \equiv 1 \pmod 4 \\ 1, \sqrt{A\varepsilon_0\sqrt{D}} & \text{if} \quad A \equiv 3 \pmod 4). \end{cases}$$

For $B = 9$, $K = k(\sqrt{2A\varepsilon_0\sqrt{D}})$ by Lemma 2, so by Theorem 2, a basis is 1, $\sqrt{2A\varepsilon_0\sqrt{D}}$.

2. $D = 1189 = 29 \cdot 41$; $\varepsilon_0 = \frac{1}{2}(25689 + 745\sqrt{1189})$; $2^{\delta/2}(r + i) = 25689 + 2i = (27 - 4i)^2 (33 + 10i)$. By Theorem 1, $K = k(\sqrt{A(1189 + B\sqrt{1189})})$ has an integral basis over k for $B = 10$ and 33, but not for $B = 17$ and 30. For $B = 10$, $K = k(\sqrt{A\varepsilon_0\sqrt{D}})$ by Lemma 2, so by Theorem 2 a basis is

$$\begin{cases} 1, \dfrac{1}{2}\left(\dfrac{1 - \sqrt{D}}{2} + \sqrt{A\varepsilon_0\sqrt{D}}\right) & \text{if} \quad A \equiv 3 \pmod 4) \\ 1, \sqrt{A\varepsilon_0\sqrt{D}} & \text{if} \quad A \equiv 1 \pmod 4). \end{cases}$$

For $B = 33$, $K = k(\sqrt{2A\varepsilon_0\sqrt{D}})$ by Lemma 2, so by Theorem 2, a basis is 1, $\sqrt{2A\varepsilon_0\sqrt{D}}$.

3. $D = 2210 = 2 \cdot 5 \cdot 13 \cdot 17$; $\varepsilon_0 = 47 + \sqrt{2210}$; $r + i = 47 + i$. By Theorem 1, $K = k(\sqrt{A(2210) + B\sqrt{2210}})$ has an integral basis over k for $B = 47$, but not for $B = 1$, 19, 23, 29, 37, 41, and 43. For $B = 47$, $K = k(\sqrt{A\varepsilon_0\sqrt{D}})$ by Lemma 2, so by Theorem 2, a basis is 1, $\sqrt{A\varepsilon_0\sqrt{D}}$.

4. $D = 221 = 13 \cdot 17$; $\varepsilon_0 = \frac{1}{2}(15 + \sqrt{221})$ has norm $+1$, so no cyclic quartic field containing $k = Q(\sqrt{221})$ has an integral basis over k.

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